UDC 532.5

ON A CERTAIN EXTENSION OF THE PROBLEM OF THE MAXWELL TOROIDAL VORTEX

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Some analogs of the Maxwell toroidal vortex are obtained for the case in which the flow inside the vortex has two velocity components and the external flow is potential. The problem is solved on the assumption that the ratio of the torus cross section radius to that of the torus is small.

Classic solutions of problems of hydrodynamics of perfect fluid flows of the kind of Hill's spherical vortex [1] or of Maxwell's toroidal vortex [2, 3] are well known. Practical feasibility and use of flows of this type were indicated in [3]. A characteristic feacture of these solutions is their cylindrical symmetry. An extension of Hill's solution in which the external flow is potential and the velocity field is essentially three-dimensional with the azimuthal velocity component inside the spherical vortex formation in addition to the radial and axial velocity components was outlined in [4]. A solution of the problem of toroidal vortex in which the azimuthal velocity component is taken into account in addition to the axial velocity component is derived in the present paper using the concept of the perfect incompressible fluid. The external potential flow is assumed to be the same as in the problem of Maxwell's vortex.

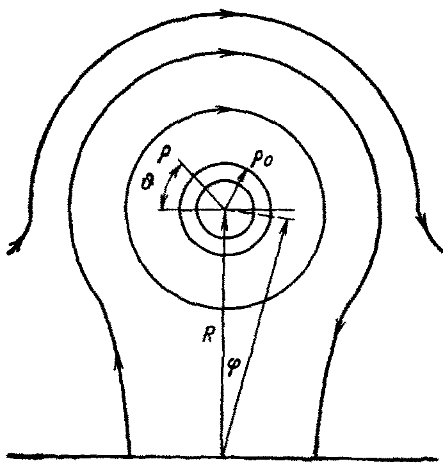

Fig. 1

Let us consider the vortex ring raising at constant velocity in a perfect incompressible fluid. Dimensions of the ring are assumed constant. Since the potential flow outside the toroidal vortex is the same as that outside the ring vortex (Maxwell's vortex), whose vorticity ω is constant over the cross section, and the axial velocity component is the only non-zero component, hence the dimensionless stream function ψ_+ of the external flow around the toroidal vortex of dimensionless radius R (the length and velocity units here are, respectively, the cross section radius a and the quantity ωa) in the system of coordinates shown in Fig. 1 is of the form [2]

$$\psi_+ (\rho) = (R/2) [\ln (8R/\rho)-1, \rho \geqslant 1 \qquad (1)$$

The coordinate origin for the radial axis ρ and axial angle ϑ is at the center of symmetry of the torus cross section. The center of symmetry of the torus cross section lies on the circumference in a plane normal to plane $\rho\vartheta$ whose position is dertermined by the azimuthal angle φ . Owing to the axial symmetry of the problem any point of the circumference can be taken as point $\varphi = 0$.

In conformity with (1) and [3, 5, 6] a region is formed around the toroidal vortex in which the fluid rotates around it and moves with it.

The equations and boundary conditions for the stream function $\psi(\vartheta, \rho)$ of the axisymmetric vortex flow of perfect incompressible fluid inside the torus is of the form [7]

$$\frac{\partial}{\partial \rho}\left(\frac{\rho}{R+\rho \sin \vartheta} \frac{\partial \psi}{\partial \rho}\right)+\frac{\partial}{\partial \vartheta}\left(\frac{1}{\rho(R+\rho \sin \vartheta)} \frac{\partial \psi}{\partial \vartheta}\right)+ \tag{2}$$

$$\frac{\rho \Phi(\psi) \Phi'(\psi)}{R+\rho \sin \vartheta}+\rho(R+\rho \sin \vartheta) F'(\psi)=0$$

$$\rho = 1, \quad \psi = \psi_+, \quad \partial\psi / \partial\rho = \partial\psi_+ / \partial\rho, \quad v_\varphi = 0$$

$$\rho = 0, \quad \psi < \infty$$

Stream functions $\psi(\vartheta, \rho)$ and $\partial\psi(\vartheta, \rho) / \partial\vartheta$ must also satisfy the conditions of periodicity with respect to ϑ. Functions $\Phi(\psi)$ and $F(\psi)$ represent circulation and the Bernoulli integral, respectively. Boundary conditions define the impenetrability and the equality of axial and azimuthal velocity components of the internal and external flows and the condition of absence of singularities at the coordinate origin.

The problem had not been analyzed in a general form even for $\Phi(\psi) = \text{const}$ and the simplest distributions $F(\psi)$. We shall investigate the flow inside a toroidal vortex for two different forms of dependence of circulation $\Phi(\psi)$ and of the Bernoulli integral on the stream function

$$\Phi_1(\psi) = k_1\psi + c_1, \quad F_i(\psi) = A_i\psi + A_i^\circ \leqslant 0 \quad (i = 1, 2) \tag{3}$$

$$\Phi_2(\psi) = [(k_2^2\psi / 2 + c_2\psi + b_2)^2]^{1/2}$$

$$k_i, c_i, A_i, A_i^\circ, \quad b_2 = \text{const}$$

The interdependence of constants c, k, A, and b_2 is determined below.

Since $\rho \sin\vartheta / R \ll 1$, the substitution $\psi(\vartheta, \rho) = \sqrt{R + \rho\sin\vartheta}\, Y(\vartheta, \rho)$ after rejection of terms of second order of smallness reduces Eq. (2) to the form

$$\frac{\partial^2 Y}{\partial \rho^2}+\frac{1}{\rho} \frac{\partial Y}{\partial \rho}+\frac{1}{\rho^2} \frac{\partial^2 Y}{\partial \vartheta^2}+\left[k_i^2-\frac{3}{4} \frac{1}{R^2}\right] Y+ \tag{4}$$

$$k_i c_i R^{-1/2} + AR^{3/2} = 0$$

whose solution is sought in the form $Y(\vartheta, \rho) = f(\rho)$. Hence the solutions of Eq. (4) with boundary conditions (2) and the dependence of A_i, c_i, and b_2 on k_i for the two indicated distributions of $\Phi(\psi)$ and $F(\psi)$ are of the form

$$\psi(\rho)=\frac{R}{2\eta} \frac{J_0(\eta_i \rho)}{J_1(\eta_i)}+\frac{R}{2}(\ln 8R-1)-\frac{R}{2\eta} \frac{J_0(\eta_i)}{J_1(\eta_i)} \tag{5}$$

$$c_1 = -(1/2)R(\ln 8R - 1)k_1, \quad \eta_i = [k_i^2 - 3 / (4R^2)]^{1/2}$$

$$A_1=\frac{1}{R}\left\{\left[\frac{J_0(\eta_1)}{2\eta_1 J_1(\eta_1)}-\frac{1}{2}(\ln 8R-1)\right] \frac{4k_1^2R^2-3}{4R^2}+\right.$$

$$\left.\frac{k_1^2}{2}(\ln 8R-1)\right\}$$

$$A_2 = \eta_2^2 (2R)^{-1} \{J_0 (\eta_2)[\eta_2 J_1 (\eta_2)]^{-1} - \ln 8R + 1\} - k_2 c_2 R^{-2}$$
$$b_2 = -(R/2)(\ln 8R - 1)[(Rk_2^2/4)(\ln 8R - 1) + c_2]$$

where $(J_n(z)$ is the Bessel function.

A flow whose circulation and Bernoulli integral are of the form (3) and the constants k_i, c_i, A_i, and b_2 are defined by (5) can, thus, be present in a toroidal vortex. The second equality of system (5) implies that in a ring vortex the flow defined by functions $\Phi(\psi)$ and $F(\psi)$ of the form $F(\psi) = A\psi + A_0$ and $\Phi(\psi) = k\psi$ is not possible. Setting in Eqs. (5) $A_i = 0$ we obtain expressions that define a uniform helical flow in the toroidal vortex, when k_1 can only have diescrete values. The formulas for velocity components within the torus in terms of ψ are of the form

$$v_{i\rho} = 0, \quad v_{i\vartheta} = -\frac{1}{R}\frac{\partial\psi}{\partial\rho} = \frac{1}{2}\frac{J_1(\eta_i\rho)}{J_1(\eta_i)} \tag{6}$$
$$v_{1\varphi} = (k_1\psi + c_1)R^{-1}$$
$$v_{2\varphi} = [2(k_2^2\psi/2 + c_2\psi + b_2)]^{1/2}R^{-1} \quad (i = 1, 2)$$

Let us compare the vorticity distribution of the considered here flow inside the torus with the vorticity of flow determined in [2]. The vortex vector components of flow (6) are of the form

$$\omega_{1\rho} \approx 0, \quad \omega_{i\varphi} = \eta_i J_0(\eta_i\rho)/(2J_1(\eta_i))$$
$$\omega_{1\vartheta} = k_1 J_1(\eta_1\rho)/[2J_1(\eta_1)]$$
$$\omega_{2\vartheta} = (k_2^2\psi + c_2)k_2 J_1(\eta_2\rho)\{2J_1(\eta_2)[2(k^2\psi^2/2 + c_2\psi + b_2)]^{1/2}\}^{-1}$$

In a torus inside which the flow has only the single axial velocity component the vortex components are of the form

$$\omega_\rho \approx 0, \quad \omega_\vartheta = 0, \quad \omega_\vartheta = 1$$

Thus the vortex vector of the considered flow at each point of surface $\rho = \text{const}$ lies in the plane tangent to it at that point and, unlike the flow in [2] has two nonzero components ω_ϑ, and ω_φ. We calculate the flow inside the torus using (5), (6) and the tables in [8]. Distribution of the azimuthal velocity component along the ρ axis is shown in Fig. 2 for $R = 10$ and $\eta = 1$. The azimuthal velocity component reaches its maximum $v_\varphi = 0.34$ at point $\rho = 0$. Streamlines for equally spaced values of $\psi(\rho)$ are concentric circiles. The maximum of $\psi(\rho)$ is reached at point $\rho = 0$ to which streamlines contract. Point $\rho = 0$ is not critical, since at it the azimuthal velocity component does not vanish.

Let us consider a ring vortex inside which the flow is uniformly helical, i. e. functions $\Phi(\psi)$ and $F(\psi)$ are of the form $\Phi(\psi) = k_1\psi + c_1$ and $F(\psi) = A_0$. When the torus radius is fixed k_1 can only have discrete values that correspond to roots of the third equation of system (5). The dependence of that root on the torus radius R is shown in Fig. 3. When $A_1 = 0$ the roots of the third equation of system (5) monotonicaly decrease with increasing radius R, and when $R \to \infty$ the first root asymptotically tends to $k = 2.404$, the second to 5. 52, the third to 8.653, and so on. These values are roots of the transcendental equation $J_0(\eta) = 0$.

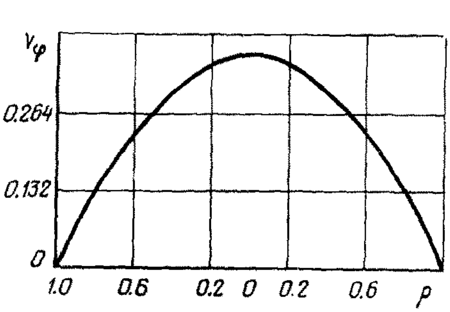

Fig. 2

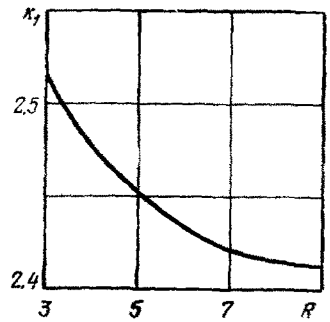

Fig. 3

In the considered case the maximum deviation of rcots of the third equation of system (5) from their asymptotic values are insignificant and decrease with increasing k_1. Thus, if k_1 is the n-th root of the third equation of system (5), the flow inside the toroidal vortex divide in n vortices whose axial velocity components are oriented in the same direction. Let us consider the case in which k_1 corresponds to $n = 4$. In that case there are four vortex formations inside the vortex ring, which lie in regions.

$$0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \vartheta \leqslant 2\pi, \quad 0 \leqslant \rho \leqslant k_1^* \,/\, k_4^*$$
$$0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \vartheta \leqslant 2\pi, \quad k_1^* \,/\, k_4^* \leqslant \rho \leqslant k_2^* \,/\, k_4^*$$
$$0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \vartheta \leqslant 2\pi, \quad k_2^* \,/\, k_4^* \leqslant \rho \leqslant k_3^* \,/\, k_4^*$$
$$0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \vartheta \leqslant 2\pi, \quad k_3^* \,/\, k_4^* \leqslant \rho \leqslant 1$$

where k_1^*, k_2^*, k_3^*, and k_4^* are roots of the third equation of system (5) with fixed radius R and $A_2 = 0$.

Note that the azimuthal velocity component and the vorticity components of the ring vortex with uniform helical flow decrease, in conformity with the second equality of system (5) and expressions (6) and (7), with increase of the torus radius, while the axial velocity component varies only slightly.

Let us now consider a toroidal vortex of radius R consisting of two vortex formations, one of which in the space between the tori defined by $1 \leqslant \rho \leqslant \rho_0$, $0 \leqslant \vartheta \leqslant 2\pi$, and $0 \leqslant \varphi \leqslant 2\pi$ and $\rho_0 \geqslant \rho \geqslant 0$, $0 \leqslant \vartheta \leqslant 2\pi$, and $0 \leqslant \varphi \leqslant 2\pi$ which we shall call the internal and external vortices, respectively. We shall investigate the flow inside these vortices assuming that the dependence of $\Phi(\psi)$ and $F(\psi)$ on the stream function is of the form $\Phi(\psi) = k\psi + c$ and $F(\psi) = A_0 + A_1\psi$, respectively, and that constants A, A_0, k, and c may, generally speaking, be different in the internal and external vortices. We denote these constant in the external vortex by subscript 1 and in the internal one by subscript 2. The solution of problem (2), (3) for the external vortex (the condition of absence of singularities at the coordinate origin is not required in this case) is of the form

$$\begin{aligned}\psi_1(\rho) = {} & [(R/2) - BR^{1/2}\eta_1 N_1(\eta_1)]J_0(\eta_1\rho)/[\eta_1 J_1(\eta_1)] + \\ & BR^{1/2}N_0(\eta_1\rho) - 4(k_1c_1 + A_1R^2)R^2/(4k_1^2R^2 - 3) \\ B = {} & \left\{\left[\frac{R}{2}(\ln 8R - 1) + \frac{4R^2(k_1c_1 + A_1R^2)}{4k_1^2R^2 - 3}\right]J_1(\eta_1) - \right. \\ & \left.\frac{R}{2\eta}J_0(\eta_1)\right\}R^{1/2}[N_0(\eta_1)J_1(\eta_1) - N_1(\eta_1)J_0(\eta_1)]^{-1} \\ c = {} & -(R/2)(\ln 8R - 1)k_1\end{aligned} \tag{7}$$

where $J_n(z)$ and $N_n(z)$ are, respectively, the Bessel and Neumann functions, and constant A_1 is arbitrary.

Let us now consider the flow in the internal vortex whose stream function $\psi_2(\rho)$ is determined by the solution of Eq. (2) with the boundary conditions

$$\begin{aligned}& \psi_2(\rho_0) = \psi_1(\rho_0), \quad \partial\psi_1(\rho_0)/\partial\rho = \partial\psi_2(\rho_0)/\partial\rho \\ & k_1\psi_1(\rho_0) + c_1 = k_2\psi_2(\rho_0) + c_2, \quad \psi_2(0) < \infty \\ & \psi_2(\rho) = -M[J_0(\eta_2\rho) - J_0(\eta_2\rho_0)]/[\eta_2 J_1(\eta_2\rho_0)] + N \\ & M = [-(R/2) + BR^{1/2}\eta_1 N_1(\eta_2)]J_1(\eta_1\rho_0)/J_1(\eta_1) - \\ & \quad BR^{1/2}\eta_1 N_1(\eta_1\rho_0), \quad N = \Psi_1(\rho_0), \quad \eta_2 = [k_2^2 - 3/(4R^2)]^{1/2} \\ & A_2 = \left[-N - \frac{MJ_0(\eta_2\rho_0)}{\eta_2 J_1(\eta_2\rho_0)}\right]\frac{4k_2^2R^2 - 3}{4R^2} - k_2N + \\ & \quad \frac{R}{2}(\ln 8R - 1) - \frac{k_2MJ_0(\eta_2\rho_0) + MJ_0(\eta_2\rho_0)}{\eta_2 J_1(\eta_2\rho_0)} + Nk_1\end{aligned} \tag{8}$$

Setting in formulas (7) and (8) $A_1 = 0$ and $A_2 = 0$ we obtain the vortex formation with uniform helical flow in the external and internal vortices, respectively.

We shall now derive a toroidal vortex formation in whose external flow is defined by functions $\Phi(\psi)$ and $F(\psi)$ of the form (3), while the flow in the internal vortex has only a single axial velocity component. In this case an additional condition that the azimuthal velocity component must be zero at surface $\rho = \rho_0$ must be imposed on the flow in the external vortex, and then A_1 is related to k_1 by the formula

$$\begin{aligned}A_1 = {} & Rk_1 2^{-1}\{\{[(\ln 8R - 1)J_1(\eta_1) + J_0(\eta_1)\eta_1^{-1}] \times \\ & J_1^{-1}(\eta_1)[X(\eta_1, \rho_0) - J_0\eta_2^{-1}]\}Y^{-1}(\eta_1, 1) + \\ & R2^{-1}(\ln 8R - 1)\}(4k^2R^2 - 3)(4R^2)^{-1} - k_1^2R2^{-1}(\ln 8R - 1)\{Y(\eta, \\ & \rho_0)Y^{-1}(\eta_1, 1) - 1\} X(\eta_1, \rho_0) = J_1(\eta_1)N_0(\eta_1\rho_0) + J_0(\eta_1\rho_0)N_1(\eta_1) \\ & Y(\eta_1, \rho_0) = J_1(\eta_1)N_0(\eta_1\rho_0) - J_0(\eta_1\rho_0)N_1(\eta_1)\end{aligned}$$

Setting in (8) $k_2 = 0$ for the stream function in the internal vortex we obtain

$$\psi_2(\rho) = M\,[I_0(\zeta\rho) - I_0(\zeta\rho_0)]\,/\,[\zeta I_1(\zeta\rho_0)] + N$$
$$\zeta = \sqrt{{}^3/_4}R^{-1}, \quad A_2 = {}^3/_4\,[N - MI_0(\zeta\rho_0)I_1^{-1}(\zeta\rho_0)]R^{-2}$$

where $I_0(z)$ and $K_0(z)$ are Bessel functions of imaginary argument.

Let us now consider the vortex formation where, unlike the previous case, the flow in the outer vortex is defined by a single axial velocity component, while in the inner vortex it is defined by functions $\Phi(\psi)$ and $F(\psi)$ of the form (3) and has two velocity components. The expression for the stream function of the outer vortex which satisfies boundary conditions (2) is of the form

$$\psi_1^*(\rho) = [R^{1/2}c_2k_1(\zeta) - R\,(2\zeta)^{-1}]I_0(\zeta\rho)I_1^{-1}(\zeta) + c_2K_0(\zeta\rho) + {}^4/_3AR^4]$$
$$c_2 = \{I_1(\zeta)[2^{-1}R^{1/2}(\ln 8R - 1) - {}^4/_3AR^{7/2}] + R^{1/2}I_0(\zeta) \times [2\zeta I_1(\zeta)]^{-1}\}[I_0(\zeta)K_1(\zeta) - K_0(\zeta)I_1(\zeta)]^{-1}$$

where constant A_1 is arbitrary.

The stream function of the inner vortex, which satisfies boundary conditions (8) is of the form

$$\psi_2 = N^* + M^*\,[J_0(\eta_2) - J_0(\eta_2\rho)][\eta_2J_1(\eta_2)]^{-1}$$
$$A_2 = -\{N^* + M^*J_0(\eta_2)[\eta_2J_1(\eta_2)]^{-1}\}\eta_2^2 + k_2^2N^*$$
$$M^* = \{R^{1/2}c_2K_1(\zeta)I_1^{-1}(\zeta) - R\,[2\zeta I_1(\zeta)]^{-1}\}I_1(\zeta\rho_0) - c_2K_1(\zeta\rho_0),$$
$$N^* = \psi_1^*(\rho), \quad c_2 = -k_2N^*$$

Hence an inner vortex can exist only then when to each fixed k_2 corresponds specific values of A_i and c_i defined by formulas (8).

Finally, let us determine the velocity of the vortex ring whose inner flow has two velocity components, and circulation and the Bernoulli integral are specified by functions (3). We introduce in the cylindrical coordinate system the quantity z_0

$$z_0 = \int |\operatorname{rot}\mathbf{v}|\,\rho^2 z\,dV \Big/ \int |\operatorname{rot}\mathbf{v}|\,\rho^2 dV$$

where the integration is carried out over the whole volume of the ring. Passing to the system of coordinates $(\rho, \vartheta, \varphi)$ shown in Fig. 1, differentiating both sides with respect to t, and taking into account that $\rho d\vartheta / dt = v_\vartheta$ and $d\rho / dt = v_\rho \approx 0$ we obtain

$$\frac{dz_0}{dt} = \frac{1}{2R}\int_0^1 |\operatorname{rot}\mathbf{v}|\,v_\vartheta\rho^2 d\rho \Big/ \left(\int_0^1 |\operatorname{rot}\mathbf{v}|\,\rho^2 d\rho\right) \tag{9}$$

The quantity dz_0 / dt is the velocity of motion of the vortex ring. The axial velocity component is determined by (6), and the expressions for $|\operatorname{rot}\mathbf{v}|$ are of the form

$$|\operatorname{rot} \mathbf{v}|_1 = \{[k_1^2 J_1^2(\eta_1\rho) + \eta_1^2 J_0^2(\eta_1\rho)] / [4J_1^2(\eta_1)]\}^{1/2}$$
$$|\operatorname{rot} \mathbf{v}|_2 = \left\{\frac{\eta_2^2 J_0^2(\eta_2\rho)}{4J_1^2(\eta_2)} + \frac{(k_2^2\psi + c_2^2)\,k_2^2 J_1^2(\eta_2\rho)}{4J_1(\eta_2)\,[2k_2^2\psi^2 + c_2\psi + b_2]}\right\}^{1/2} \tag{10}$$

The first equality of system (10) defines the modulus of vorticity in the vortex ring inside which the flow is defined by functions $\Phi_1(\psi)$ and $F_1(\psi)$ of the form $\Phi_1(\psi) = k_1\psi + c_1$ and $F_1(\psi) = A_1\psi + A_1^\circ$; the second of these equalities defines it in the vortex ring inside which the flow is determined by functions $\Phi_2(\psi)$ and $F_2(\psi)$ of the form $F_2(\psi) = A_2\psi + A_2^\circ$ and $\Phi(\psi) = (k_2^2\psi^2 / 2 + c_2\psi + b_2)^{1/2}$. The dependence of the velocity of the vortex ring rise on the torus radius R is shown in Fig. 4 by curve 1 for $\eta_1 = 1$ and functions $\Phi_1(\psi)$ and $F_1(\psi)$ of the form $\Phi_1(\psi) = k_1\psi + c_1$ and $F(\psi) = A_1^\circ + A_1\psi$, and the velocity of rise determined by formula (9). For comparison, the velocity of rise of a vortex ring inside which the flow has only the axial velocity component [2] is shown in Fig. 4 by curve 2. The difference between the rise velocities in the two cases is small which is related to the decrease of the azimuthal velocity component with increased torus radius, while the axial velocity components vary only insignificantly. Hence for large torus radii the azimuthal velocity component is small and ring vortices differ only slightly. It will be seen from Fig. 4 that the rise velocities of both ring vortices monotonically decrease with increased torus radius; when $R = 14.9$ these velocities are equal, while for $R < 14.9$ the velocity of rise of the ring vortex with an azimuthal twist is higher than without it.

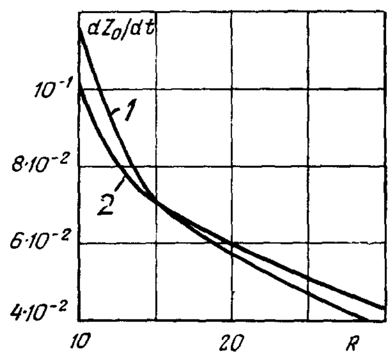

Fig. 4

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